ON CONSTRUCTION OF A SERIES OF BALANCED INCOMPLETE BLOCK DESIGNS

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INTRODUCTION

While going through the table of Incomplete Block Designs with replications from 11 to 15 presented in the table of Statisticians and Biometricians by Fisher and Yates (1963) it was found that the design

$$v=14,\ b=26,\ k=7,\ r=13,\ \lambda=6$$

has not been presented in the table. In an effort to search out a method of construction of the design we could get a general solution of the following series of balanced incomplete block designs :

$$v=p+1,\ b=2p,\ k=(p+1)/2,\ r=p,\ \lambda=(p-1)/2$$

where p i an odd prime or an odd prime power. The solution can be obtained through the following theorem :

THEOREM I

When in a Balanced Incomplete Block Design the number of treatments p is an odd prime or an odd prime power, the two initial blocks formed of the even powers of a primitive element of G.F. (p) taken (*i*) once with zero and (*ii*) again with ∞ give a Balanced Incomplete Block Design when developed. The parameters of the design are

$$v=p+1,\ b=2p,\ k=(p+1)/2,\ r=p,\ \lambda=(p-1)/2$$

Proof. Let x stand for a primitive element of G.F. (p). Let us consider, the following initial block formed of the even powers of the primitive element x

$$x^0 , x^2 , x^4 \ldots\ldots\ldots x^{2m} \ldots\ldots\ldots x^{2m+2s} \ldots\ldots\ldots x^{2(p-3)/2}$$

The difference between the element x^{2m} and any other at a distance s in the above initial block is given by

$$x^{2m+2s} - x^{2m} = x^{+2m} \ (x^{2s}-1)$$

$$\left(\begin{array}{l} m=0, \ 1,\ldots \frac{p-3}{2} \\ s=1, \ 2,\ldots\ldots\ldots p-5/2 \end{array} \right)$$

$$x^{p-1}=1$$

Let us now consider a particular value of a difference, say x^{2q}. This value, x^{2q}, will occur in all the $(k-1)(k-2)$ differences, r_1 times, where r_1 stands for the number of different solutions of the equations :

$$x^{2m} \ (x^{2s}-1)=x^{2q} \qquad \ldots(1)$$

for varying m and s. Evidently the number of solutions is equal to the number of times $x^{2s}-1$ is equal to an even power of x.

Let us now consider the difference x^t-1 where t varies from $1, 2,\ldots\ldots, p-2.$ Evidently x^t-1 can take all the non-zero values excepting $\frac{p-1}{x^2}$, as $\frac{p-1}{x^2}$ is equal to -1. When t is even, let there be x_1 values of t such that in each case x^t-1 is equal to an even power of x and y_1 values of t such that in each case x^t-1 is equal to odd power of x. Hence $x_1+y_1=(p-3)/2$ as an even t can have $(p-3)/2$ non zero values. Evidently, we are interested to find out the value of x_1 as this gives the number of solutions of the equation at (1).

Whe *t is odd* let there be x_2 values of t such that x^t-1 is equal to an even power of x and y_2 values are equal to an odd power of x. Then $x_2+y_2=(p-1)/2$. Again $x_1+x_2=(p-3)/2$ as this is equal to total number of even powers of x excepting zero.

As p is an odd prime power, it can be either of the form $4n+1$ or $4n+3$, wher n is an integer.

Case I. When $p=4n+1$, so that $(p-1)/2=2n$ is equal to an even number.

We also know that

$$x^{\frac{p-1}{2}}=-1 \text{ i.e. } x^{2n}=-1$$

Again $\qquad x^{p-1}=1 \quad$ i.e. $\quad x^{4n}=1$

Let $\qquad x^{2r+1}-1=x^{2n}$

where r is any integer, then it can be shown that $x^{4n-2r-1}$ is equal to an odd power of x.

For, $x^{4n-2r-1}-1=\dfrac{1}{x^{2r+1}}-1$

$$=(1-x^{2r+1})/x^{2r+1}$$

$$=+x^{2n}/x^{2r+1}$$

$$=-x^{2(n-r)-1}$$

$$=x^{2n}(x)^{2(n-r)-1}$$

$$=x^{2(n+n-r)-1}$$

$$=\text{an odd power of } x$$

Thus if $x^{2r+1}-1$ is equal to an even power of x, then $x^{4n-2r-1}-1$ is equal to an odd power of x. Hence among the difference $x^{2r+1}-1$ for all values of r there will be an equal number of odd and even powers of x. This means $x_2=y_2$. We have thus the following equations.

$$x_1+y_1=(p-3)/2$$
$$x_2+y_2=(p-1)/2$$
$$x_1+y_2=(p-3)/2$$
$$x_2=y_2$$

Solving these equations we get

$$x_1=(p-5)/4,\ y_1=(p-1)/4$$

Thus a particular difference x^{2q} can occur among the difference $x^{2m}(x^{2s}-1)$, $(p-5)/4$ times when $p=4n+1$. This result holds for any even power of x. Hence among the difference every element which is equal to an even power of x will occur $(p-5)/4$ times.

Next any difference x^{2m} $(x^{2s}-1)=x^{2q+1}$ an odd power of x, will occur as many times as $x^{2s}-1$ is equal to an odd power of x. This number is equal to $y_1=(p-1)/4$. Thus every difference which is equal to an odd power of x will occur $(p-1)/4$ times when $p=4n+1$.

Hence the differences which are equal to odd powers of x, occur one time more than the difference which are equal to even powers of x.

Again among the difference$=(O-x^{2m})$ each even power, x^{2q}, will occur twice as $-x^{2m}=x^{2m+2n}$. So if the initial block formed of the even powers of x be taken twice, once with the element zero and again with the element infinity each non zero element, will occure $(p-1)/2$ times in the differences obtained from the two blocks.

If again we form the initial block with the odd powers of x, it can be shown that any difference is equal to an even power of x, will occur $(p-1)/4$ times and any difference equal to an odd power of x will occur $(p-5)/4$ times. Thus the two initial blocks, one formed of the odd powers of x with zero and the other formed of the odd powers of x with ∞ will also generate the BIB Design.

Case 2. When $p=4n+3$. It can be easily shown that $x^{2m}-1$ will be in λ cases equal to an even power of x and in other λ cases equal to an odd power of x. Thus any element will occur the same number of times in the difference, a fact which is well known. It is also well known that among the differences $\pm$ $(O-x^{2m})$ all the non-zero elements occur the same number of times. These two facts ensure that the two initial blocks, one formed of the even powers of the primitives elements along with zero and the other even powers of the primitive elements along with infinity will give a BIB design with the same parameters as indicated earlier.

THEOREM 2

When in a Balanced Incomplete Block Design the number of treatments p is an odd prime or an odd prime power of the form $4n+1$ the two initial blocks—One formed of the even powers of the primitive elements of G.F. (p) and the other formed of the odd powers of primitive elements will give a BIB design when developed. The parameters of the design will be

$$v=p,\ b=2p,\ k=(p-1)/2,\ r=p-1,\ \lambda=(p-3)/2$$

Proof. We have seen that when the initial block is formed of the even powers of the primitive element, any difference equal to an even powers of x will occur $(p-5)/4$ times and any difference equal to an odd power of x will occur $(p-1)/4$ times, among all the differences formed of the elements in the initial blocks. It was also indicated that when the initial block is formed of the odd powers of the primitive elements, any difference which is equal to an even power of x will occur $(p-1)/4$ times and any difference which is equal to an odd power of x will occur $(p-5)/4$ times among the difference obtainable from the initial blocks. Thus these two initial blocks will form a BIB design with $\lambda=(p-1)/4+(p-5)/4=(p-3)/2$.

As an illustration we shall construct the following two designs:

(*i*) $v=14$, $b=26$, $k=7$, $r=13$ $\lambda=6$

(*ii*) $v=13$, $b=26$, $k=6$, $r=12$ $\lambda=5$

As the primitive root of 13 is 2, and the difference of even powers of 2 are equal to 1, 4, 3, 12, 9, 10

The following two initial blocks

(*i*) 0, 1, 3, 4, 9, 10, 12

(*ii*) ∞, 1, 3, 4, 9, 10, 12

will generate the first design when developed. This design is missing in Fisher and Yates tables XIX : 2.

The second design can be generated from two initial blocks – One formed of the even powers of 2 and the other formed of the odd powers of 2. These two blocks are

(*i*) 1, 3, 4, 9, 10, 12

(*ii*) 2, 5, 6, 7, 8, 11

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REFERENCE

1. Fisher, R.A. and Yates, F. (1963) ; Statistical Tables for Biological, Agricultural and Medical Research, Sixth Edition, Oliver and Boyd Ltd., Edinburg.